Unified Quiz 3S
March 23, 2007

ID number (last four digits) SOLUTION

Quiz Instructions:

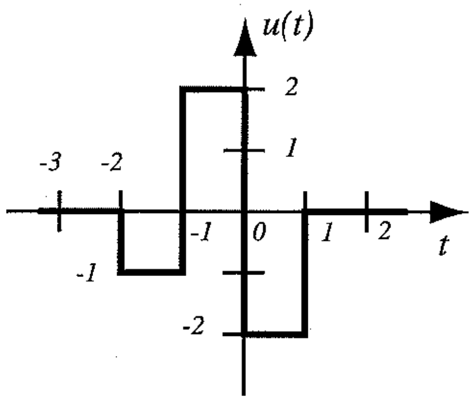
- One 8.5 x 11 crib sheet (both sides) allowed. No books or notes allowed.
- Calculators are not needed, and may not be used.
- Put the last 4 digits of your ID on each page of the exam.
- Read all questions carefully.
- Do all work for each problem on the FRONT of the pages provided.
- Show intermediate results.
- Explain your work — don't just write equations. Any problem (except multiple choice) without an explanation can receive no better than a "B" grade.
- Partial credit will be given, but only when the intermediate results and explanations are clear.
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Show appropriate units with your final answers.
- Box your final answers.

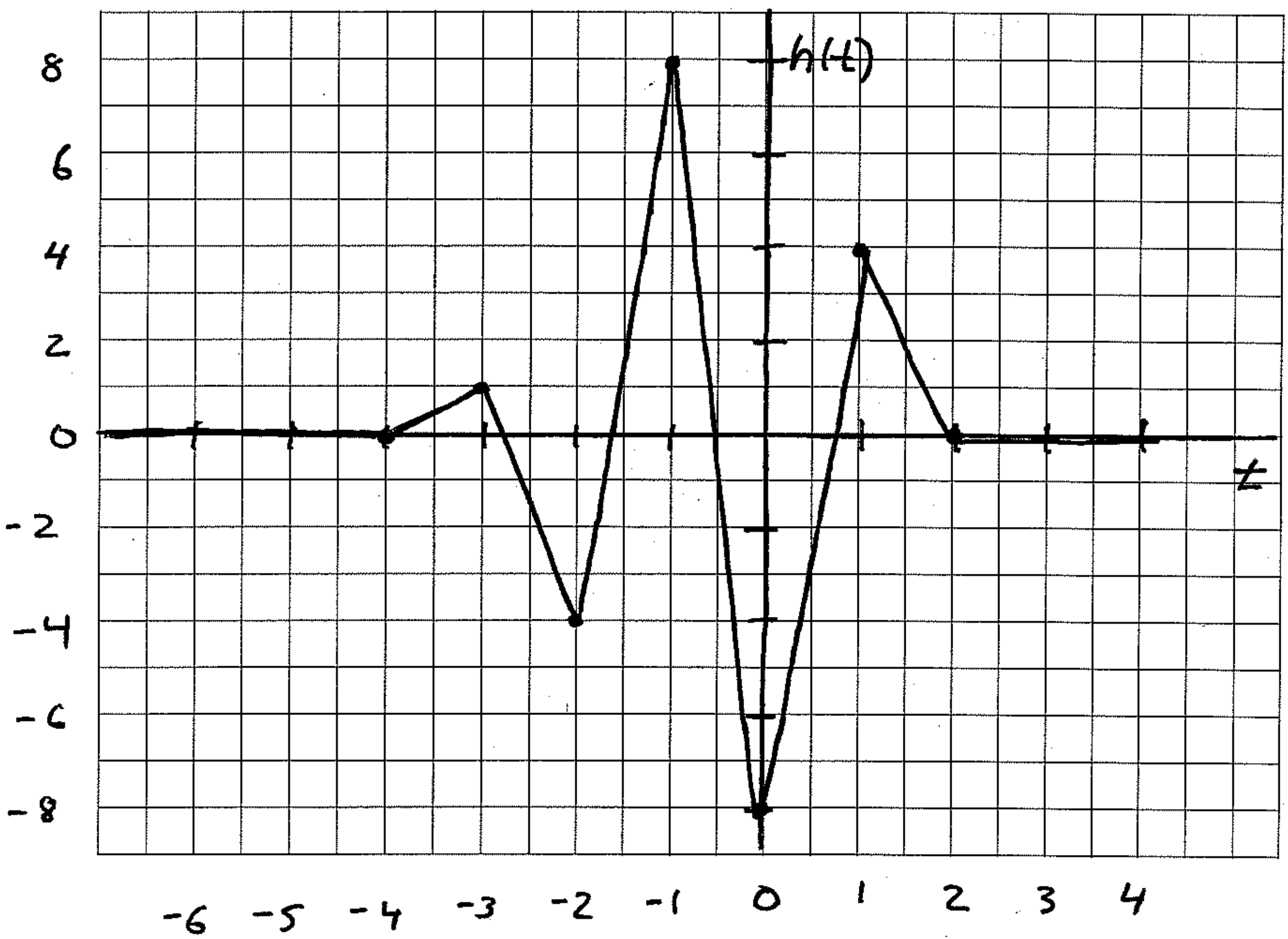
Exam Scoring:

Problem 1 (20 points)	
Problem 2 (10 points)	
Problem 3 (20 points)	
Problem 4 (20 points)	
Total (70 points)	
Total (percent)	

Problem 1 (20 points) ID number (last four digits) SOLUTION

The signal $u(t)$ is shown in the graph below. Sketch in the grid below, as accurately as possible, the signal $h(t) = u(t) * u(t)$. Be sure to label the axes of the grid. The grid squares *do not* have to represent 1 unit — you can chose the units as appropriate to plot the result. Be sure to label the axes of the grid. Explain your reasoning on the pages that follows.

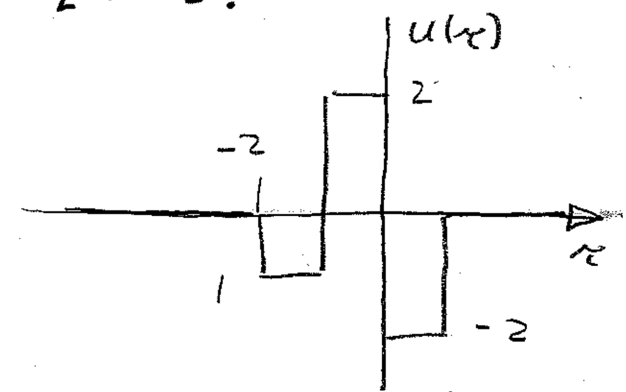


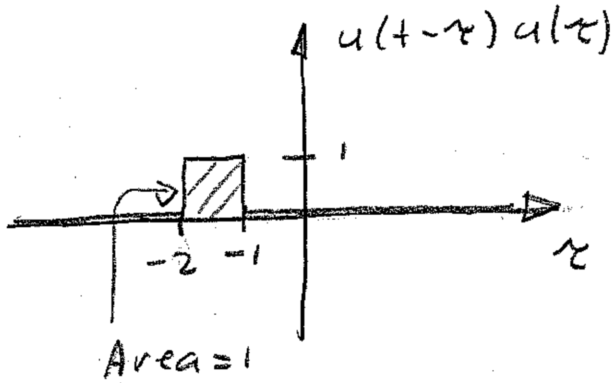


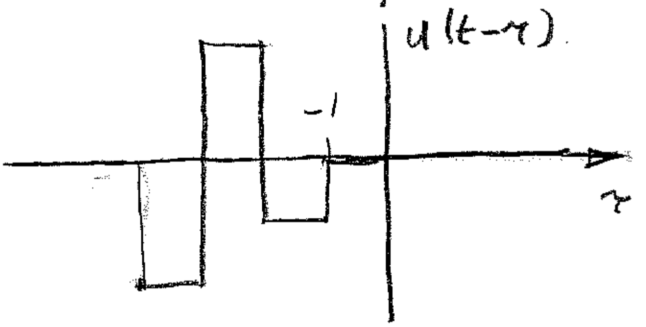
To convolve, use graphical techniques, i.e., flip and slide. Since $u(t)$ is piecewise constant, $u * u$ will be piecewise linear and continous. So by finding $y(t)$ at the "kinks," can accurately plot the curve of $y(t)$.

There is no overlap of $u(\tau)$ and $u(t-\tau)$ (that is, where both functions are nonzero) for $t \le -4$ or $t \geq 2$, so $y(t) = 0$ for these t. For $-4 < t < 2$, look at specific values of t:

$t = -3$:

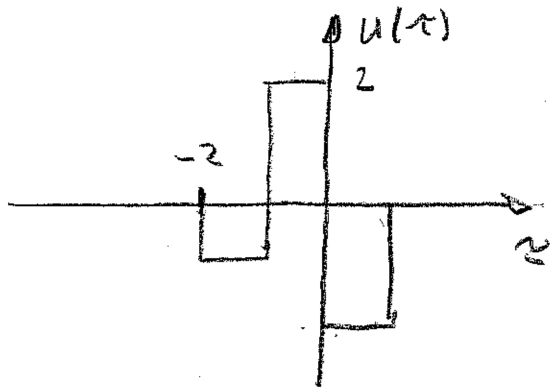


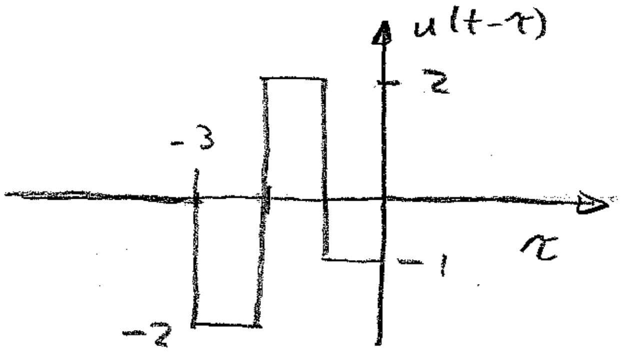


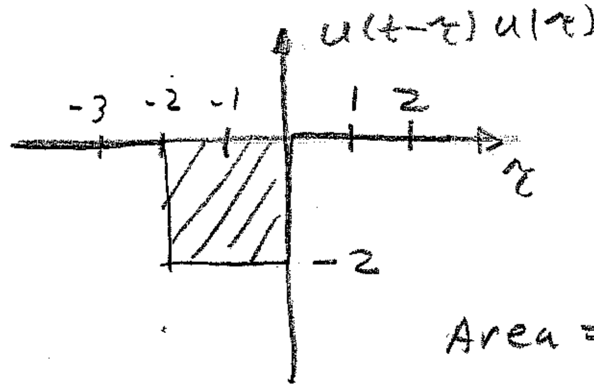


For $t = -3$, $y(-3) = 1$.

For $t = -2$:







Area = −4

$\Rightarrow y(-2) = -4$

Continuing in this fashion, we have

t	$y(t)$
−4	0
−3	1
−2	−4
−1	8
0	−8
1	4
2	0

From these values and the discussion above, we obtain the plot.

Problem 2 (10 points) **ID number (last four digits)** SOLUTION

The region of convergence of the Laplace transform of a causal signal always has the form $\text{Re}[s] > \sigma_0$, where σ_0 is a real number. As special cases, the integral may converge for all s, or there may be no region of convergence at all. Explain why the region of convergence has the form [redacted] $\text{Re}[s] > \sigma_0$.

Suppose the LT

$$G(s) = \int_0^\infty g(t)\, e^{-st}\, dt$$

converges for some $s = s_1 = \sigma_1 + j\omega_1$. Then it must be true that the integrand

$$g(t)\, e^{-st} \to 0$$

as $t \to \infty$ — otherwise, the integral wouldn't converge.

Now consider $s = s_2 = \sigma_2 + j\omega_2$, where $\sigma_2 > \sigma_1$. Compare the two integrands

$$g(t)\, e^{-s_1 t} \quad \text{and} \quad g(t)\, e^{-s_2 t}$$

The magnitudes are

$$\left| g(t)\, e^{-(\sigma_1 + j\omega_1)t} \right| = |g(t)| \left| e^{-\sigma_1 t} \right|$$

and

$$|g(t)||e^{-\sigma_2 t}|$$

Since the magnitude of a complex exponential depends only on the real part of the exponent. Therefore, the second integrand is smaller than the first by a factor

$$e^{-(\sigma_2-\sigma_1)t}$$

and since $\sigma_2 > \sigma_1$, the second integrand is always smaller (in fact exponentially smaller) than the first, and so the integral must converge for $s = s_2$. That is, if the integral converges for some $s = s_1$, it must converge for all s such that $\mathrm{Re}[s] > \mathrm{Re}[s_1]$. Finding the most negative real part of s for which the integral converges, and denoting it by σ_0, we have the r.o.c. is

$$\mathrm{Re}[s] > \sigma_0.$$

Problem 3 (20 points) **ID number (last four digits)** SOLUTION

An LTI system, G, has impulse response

$$g(t) = e^{t}\sigma(t)$$

Find the response of the system, $y(t)$, to the input

$$u(t) = 4te^{-3t}\sigma(t)$$

Hint: There is more than one way to solve this problem. Choose the method that is fastest!

The easiest way to solve is to use Laplace techniques. Since the system is LTI, we know

$$y(t) = g(t) * u(t)$$

Taking the LT of both sides,

$$Y(s) = G(s)\,U(s)$$

$g(t)$ and $u(t)$ are simple enough that we can find their LTs by direct integration, from a table, or "by inspection." The result is

$$G(s) = \frac{1}{s-1}, \qquad \mathrm{Re}[s] > 1$$

$$U(s) = \frac{4}{(s+3)^2}, \qquad \mathrm{Re}[s] > -3$$

Therefore,

$$Y(s) = \frac{4}{(s-1)(s+3)^2}, \quad Re[s] > 1$$

To inverse LT, do partial fraction expansion, using coverup method, to obtain

$$Y(s) = \frac{1/4}{s-1} - \frac{1}{(s+3)^2} + \frac{c}{s+3}$$

To find c, pick a value of s, say s = 0, so that

$$Y(0) = -\frac{4}{9} = -\frac{1}{4} - \frac{1}{9} + \frac{c}{3}$$

Solving for c, we have

$$c = -\frac{1}{4}$$

Therefore,

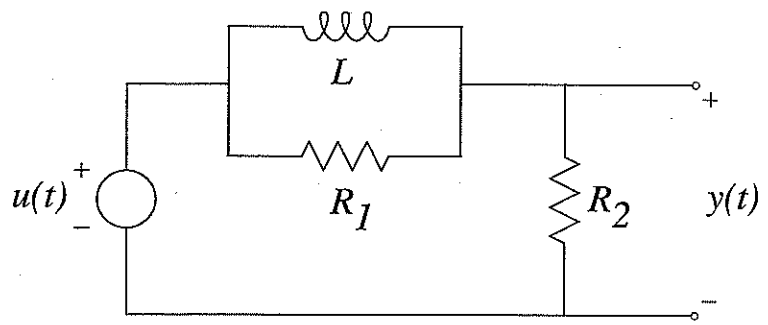
$$Y(s) = \frac{1/4}{s-1} - \frac{1}{(s+3)^2} - \frac{1/4}{s+3}, \quad Re[s] > 1$$

Using a table or by inspection,

$$y(t) = \left(\frac{1}{4}e^{t} - te^{-3t} - \frac{1}{4}e^{-3t}\right)\sigma(t)$$

Problem 4 (20 points) ID number (last four digits) SOLUTION

Consider the circuit



with component values $L = 1\text{H}$, $R_1 = 4\,\Omega$, $R_2 = 4\,\Omega$. Find the response of the circuit, $y(t)$, to the input $u(t) = e^{-t}\sigma(t)$.

Use impedance and LT techniques. The transfer function $G(s)$ can be found using impedance methods. In fact, this is a voltage divider, so

$$G(s) = \frac{R_2}{R_2 + R_1 \| Ls}$$

and

$$R_1 \| Ls = \frac{R_1 Ls}{R_1 + Ls} = \frac{4s}{s+4}$$

So
$$G(s) = \frac{4}{4 + \frac{4s}{s+4}}$$

Multiply top & bottom by $s+4$ to get

Problem 4 ID number (last four digits) SOLUTION

$$G(s) = \frac{4s + 16}{8s + 16}$$

$$= \frac{1}{2}\,\frac{s+4}{s+2}, \quad Re[s] > -2$$

The input has LT

$$U(s) = \mathcal{L}[u(t)] = \frac{1}{s+1}, \quad Re[s] > -1$$

Therefore,

$$Y(s) = G(s)\,U(s) = \frac{1}{2}\,\frac{s+4}{(s+2)(s+1)}, \quad Re[s] > -1$$

By PFE,

$$Y(s) = \frac{3/2}{s+1} - \frac{1}{s+2}, \quad Re[s] > -1$$

The inverse LT is then

$$y(t) = \left(\frac{3}{2}e^{-t} - e^{-2t}\right)u(t)$$